

Derivácie elementárnych funkcií

Konštantná funkcia

$$(c)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{c - c}{x - x_0} = \lim_{x \rightarrow x_0} \frac{0}{x - x_0} = 0$$

Mocninové funkcie

$$(x)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x - x_0}{x - x_0} = 1$$

$$(x^2)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \rightarrow x_0} (x + x_0) = x_0 + x_0 = 2x_0$$

$$(x^3)' = \lim_{x \rightarrow x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = \lim_{x \rightarrow x_0} (x^2 + xx_0 + x_0^2) = x_0^2 + x_0^2 + x_0^2 = 3x_0^2$$

$$(x^4)' = \lim_{x \rightarrow x_0} \frac{x^4 - x_0^4}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x^3 + x^2x_0 + xx_0^2 + x_0^3)}{x - x_0} = \lim_{x \rightarrow x_0} (x^3 + x^2x_0 + xx_0^2 + x_0^3) = 4x_0^3$$

$$(x^n)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + xx_0^{n-2} + x_0^{n-1})}{x - x_0} = \\ = \lim_{x \rightarrow x_0} (x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + xx_0^{n-2} + x_0^{n-1}) = n \cdot x_0^{n-1}$$

$$\left(\frac{1}{x^n}\right)' = (x^{-n})' = -n \cdot x^{-n-1} = -n \cdot x^{-(n+1)} = \frac{-n}{x^{n+1}}$$

$$\left(\sqrt[n]{x}\right)' = \left(x^{\frac{1}{n}}\right)' = \frac{1}{n} x^{\frac{1}{n}-1} = \frac{1}{n} x^{\frac{1-n}{n}} = \frac{\sqrt[n]{x^{1-n}}}{n} = \\ = \frac{1}{n} x^{-\frac{n-1}{n}} = \frac{1}{n \cdot x^{\frac{n-1}{n}}} = \frac{1}{n \cdot \sqrt[n]{x^{n-1}}}$$

príklad:

$$(3)' = 0$$

$$(-7)' = 0$$

$$(-2,897)' = 0$$

$$\left(\frac{101}{13}\right)' = 0$$

$$(\sqrt{11})' = 0$$

$$(\pi)' = 0$$

$$(x)' = 1$$

$$(2x)' = 2 \cdot (x)' = 2 \cdot 1 = 2$$

$$(11x)' = 11 \cdot (x)' = 11 \cdot 1 = 11$$

$$(-29x)' = -29(x)' = -29$$

$$\left(\frac{5}{9}x\right)' = \frac{5}{9}(x)' = \frac{5}{9}$$

$$(\sqrt{3}x)' = \sqrt{3}(x)' = \sqrt{3}$$

$$(x^2)' = 2x$$

$$(5x^2)' = 5 \cdot (x^2)' = 5 \cdot 2x = 10x$$

$$(34x^2)' = 34(x^2)' = 34 \cdot 2x = 68x$$

$$(-7x^2)' = -7 \cdot 2x = -14x$$

$$\left(-\frac{13}{12}x^2\right)' = -\frac{13}{12} \cdot 2x = -\frac{13}{6}x$$

$$(x^3)' = 3x^2$$

$$(4x^3)' = 4 \cdot (x^3)' = 4 \cdot 3x^2 = 12x^2$$

$$(22x^3)' = 22 \cdot 3x^2 = 66x^2$$

$$(-10x^3)' = -10 \cdot 3x^2 = -30x^2$$

$$(x^4)' = 4x^3$$

$$(x^6)' = 6x^5$$

$$(x^{14})' = 14x^{13}$$

$$(x^{501})' = 501x^{500}$$

$$(4x^7 - 3x^6 + 5x^4)' = (4x^7)' - (3x^6)' + (5x^4)' = 4 \cdot (x^7)' - 3 \cdot (x^6)' + 5 \cdot (x^4)' = 4 \cdot 7x^6 - 3 \cdot 6x^5 + 5 \cdot 4x^3 = 28x^6 - 18x^5 + 20x^3$$

$$(2x^{11} - 6x^9 - 3x^5 + 88)' = (2x^{11})' - (6x^9)' - (3x^5)' + (88)' = 2(x^{11})' - 6(x^9)' - 3(x^5)' + 0 = 22x^{10} - 54x^8 - 15x^4$$

$$(x^{-7})' = -7x^{-8} = -\frac{7}{x^8}$$

$$(3x^{-24})' = 3 \cdot (x^{-24})' = 3 \cdot (-24)x^{-25} = -72x^{-25} = -\frac{72}{x^{25}}$$

$$(-2x^{-98})' = -2 \cdot (x^{-98})' = -2 \cdot (-98)x^{-99} = 196x^{-99} = \frac{196}{x^{99}}$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = -1x^{-2} = -\frac{1}{x^2}$$

$$\left(\frac{5}{x}\right)' = (5x^{-1})' = 5 \cdot (x^{-1})' = 5 \cdot (-1) \cdot x^{-2} = -5x^{-2} = -\frac{5}{x^2}$$

$$\left(\frac{3}{x^2}\right)' = (3x^{-2})' = 3 \cdot (-2)x^{-3} = -\frac{6}{x^3}$$

$$\left(\frac{-8}{x^5}\right)' = (-8x^{-5})' = -8 \cdot (-5)x^{-6} = \frac{40}{x^6}$$

$$\left(\frac{-12}{x^7}\right)' = (-12x^{-7})' = -12 \cdot (-7)x^{-8} = \frac{84}{x^8}$$

$$\left(\frac{2}{x^{13}}\right)' = (2x^{-13})' = 2 \cdot (-13)x^{-14} = -\frac{26}{x^{14}}$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

$$(\sqrt[3]{x})' = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$(\sqrt[7]{x})' = \left(x^{\frac{1}{7}}\right)' = \frac{1}{7}x^{-\frac{6}{7}} = \frac{1}{7x^{\frac{6}{7}}} = \frac{1}{7\sqrt[7]{x^6}}$$

$$(\sqrt[5]{x^8})' = \left(x^{\frac{8}{5}}\right)' = \frac{8}{5}x^{\frac{3}{5}} = \frac{8}{5}\sqrt[5]{x^3}$$

$$(\sqrt[11]{x^3})' = \left(x^{\frac{3}{11}}\right)' = \frac{3}{11}x^{-\frac{8}{11}} = \frac{3}{11x^{\frac{8}{11}}} = \frac{3}{11\sqrt[11]{x^8}}$$

$$(\sqrt[7]{x^{11}})' = \left(7x^{\frac{11}{6}}\right)' = 7 \cdot \frac{11}{6}x^{\frac{5}{6}} = \frac{77}{6}\sqrt[6]{x^5}$$

$$\left(-\frac{2}{3}\sqrt[7]{x^5}\right)' = \left(-\frac{2}{3}x^{\frac{5}{7}}\right)' = -\frac{2}{3} \cdot \frac{5}{7}x^{-\frac{2}{7}} = -\frac{10}{21x^{\frac{2}{7}}} = -\frac{10}{21\sqrt[7]{x^2}}$$

$$\left(\frac{1}{\sqrt{x}}\right)' = \left(\frac{1}{x^{\frac{1}{2}}}\right)' = \left(x^{-\frac{1}{2}}\right)' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2\sqrt{x^3}}$$

$$\left(\frac{1}{\sqrt[3]{x}}\right)' = \left(\frac{1}{x^{\frac{1}{3}}}\right)' = \left(x^{-\frac{1}{3}}\right)' = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x^{\frac{4}{3}}} = -\frac{1}{3\sqrt[3]{x^4}}$$

$$\left(\frac{1}{\sqrt[12]{x}}\right)' = \left(\frac{1}{x^{\frac{1}{12}}}\right)' = \left(x^{-\frac{1}{12}}\right)' = -\frac{1}{12}x^{-\frac{13}{12}} = -\frac{1}{12x^{\frac{13}{12}}} = -\frac{1}{12\sqrt[12]{x^{13}}}$$

$$\left(\frac{1}{\sqrt[4]{x^7}}\right)' = \left(\frac{1}{x^{\frac{7}{4}}}\right)' = \left(x^{-\frac{7}{4}}\right)' = -\frac{7}{4}x^{-\frac{11}{4}} = -\frac{7}{4x^{\frac{11}{4}}} = -\frac{7}{4\sqrt[4]{x^{11}}}$$

$$\left(\frac{1}{\sqrt[13]{x^6}}\right)' = \left(\frac{1}{x^{\frac{6}{13}}}\right)' = \left(x^{-\frac{6}{13}}\right)' = -\frac{6}{13}x^{-\frac{19}{13}} = -\frac{6}{13x^{\frac{19}{13}}} = -\frac{6}{13\sqrt[13]{x^{19}}}$$

$$\left(\frac{3}{\sqrt[5]{x^6}}\right)' = \left(\frac{3}{2x^{\frac{6}{5}}}\right)' = \left(\frac{3}{2}x^{-\frac{6}{5}}\right)' = \frac{3}{2} \cdot \left(-\frac{6}{5}\right)x^{-\frac{11}{5}} = -\frac{9}{5x^{\frac{11}{5}}} = -\frac{9}{5\sqrt[5]{x^{11}}}$$

$$\left(\frac{-5}{\sqrt[9]{x^8}}\right)' = \left(\frac{-5}{x^{\frac{8}{9}}}\right)' = \left(-5x^{-\frac{8}{9}}\right)' = -5 \cdot \left(-\frac{8}{9}\right)x^{-\frac{17}{9}} = \frac{40}{9x^{\frac{17}{9}}} = \frac{40}{9\sqrt[9]{x^{17}}}$$

$$(\sin x)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\sin x - \sin x_0}{x - x_0} = \lim_{x \rightarrow x_0} \frac{2 \cos \frac{x+x_0}{2} \sin \frac{x-x_0}{2}}{x - x_0} = \lim_{x \rightarrow x_0} \frac{2 \cos \frac{x+x_0}{2} \sin \frac{x-x_0}{2}}{2 \cdot \frac{x-x_0}{2}} =$$

$$= \lim_{x \rightarrow x_0} \cos \frac{x+x_0}{2} = \cos \frac{2x_0}{2} = \cos x_0$$

$$(\cos x)' = \lim_{x \rightarrow x_0} \frac{\cos x - \cos x_0}{x - x_0} = \lim_{x \rightarrow x_0} \frac{-2 \sin \frac{x+x_0}{2} \sin \frac{x-x_0}{2}}{x - x_0} = \lim_{x \rightarrow x_0} \frac{-2 \sin \frac{x+x_0}{2} \sin \frac{x-x_0}{2}}{2 \cdot \frac{x-x_0}{2}} = \lim_{x \rightarrow x_0} -\sin \frac{x+x_0}{2} =$$

$$= -\sin \frac{2x_0}{2} = -\sin x_0$$

$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\operatorname{cotg} x)' = \left(\frac{\cos x}{\sin x} \right)' = \frac{(\cos x)' \cdot \sin x - \cos x \cdot (\sin x)'}{(\sin x)^2} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

Zložená funkcia

Funkcia je zložená, ak namiesto x -u nájdeme inú funkciu ako argument. Môže byť aj viacnásobne zložená.

pr. $\sin 2x$; $\sqrt{x-3}$; 2^{3x-7} ; $\log x^3$; $\frac{1}{x^2-7x+3}$; $\operatorname{tg} \sqrt{e^{x+3}}$

Zložené funkcie sa skladajú z viacerých funkcií (zložiek). Vnútoraná funkcia (zložka) je vždy tá, ktorú počítame ako prvú, ak chceme vypočítať funkčnú hodnotu v konkrétnom bode.

$\sin 2x$ pre $x = 27^\circ$

najprv: $2 \cdot 27^\circ = 54^\circ$

a potom: $\sin 54^\circ = 0,809$

$\sqrt{x-3}$ pre $x = 14$

najprv: $14 - 3 = 11$

a potom: $\sqrt{11} = 3,317$

2^{3x-7} pre $x = 2$

najprv: $3 \cdot 2 - 7 = 6 - 7 = -1$

a potom: $2^{-1} = 0,5$

$\log x^3$ pre $x = 4$

najprv: $4^3 = 64$

a potom: $\log 64 = 1,806 2$

$\frac{1}{x^2-7x+3}$ pre $x = 3$

najprv: $3^2 - 7 \cdot 3 + 3 = 9 - 21 + 3 = -9$

a potom: $\frac{1}{-9} = -0,1\bar{1}$

$\operatorname{tg} \sqrt{e^{x+3}}$ pre $x = 5$

najprv: $5 + 3 = 8$

potom: $e^8 = 2\,980,96$

potom: $\sqrt{2\,980,96} = 54,60$

a naposledy: $\operatorname{tg} 54,60 = 2,506$ (v radiánoch počítané)

V. $(f \circ g)'(x) = \{f[g(x)]\}' = f'[g(x)] \cdot g'(x)$

Zloženú funkciu derivujeme tak, že derivujeme vonkajšiu funkciu vo vnútornej funkcii ako bode, a vynásobíme derivovanou vnútornou funkciou.

príklad:

$$(\sin 2x)' = \cos 2x \cdot (2x)' = \cos 2x \cdot 2 = 2 \cdot \cos 2x$$

$$(\sqrt{x-3})' = \left((x-3)^{\frac{1}{2}} \right)' = \frac{1}{2} (x-3)^{-\frac{1}{2}} \cdot (x-3)' = \frac{1}{2\sqrt{x-3}} \cdot 1 = \frac{1}{2\sqrt{x-3}}$$

$$(2^{3x-7})' = 2^{3x-7} \cdot \ln 2 \cdot (3x-7)' = 2^{3x-7} \cdot \ln 2 \cdot 3 = 3 \cdot 2^{3x-7} \cdot \ln 2$$

$$(\log x^3)' = \frac{1}{x^3 \cdot \ln 10} \cdot (x^3)' = \frac{1}{x^3 \cdot \ln 10} \cdot 3x^2 = \frac{3}{x \cdot \ln 10}$$

$$\left(\frac{1}{x^2-7x+3} \right)' = \left((x^2-7x+3)^{-1} \right)' = -1 \cdot (x^2-7x+3)^{-2} \cdot (x^2-7x+3)' = \frac{-1}{(x^2-7x+3)^2} \cdot (2x-7) = \frac{7-2x}{(x^2-7x+3)^2}$$

$$\begin{aligned} (\operatorname{tg} \sqrt{e^{x+3}})' &= \frac{1}{\cos^2 \sqrt{e^{x+3}}} \cdot \left((e^{x+3})^{\frac{1}{2}} \right)' = \frac{1}{\cos^2 \sqrt{e^{x+3}}} \cdot \frac{1}{2} (e^{x+3})^{-\frac{1}{2}} \cdot (e^{x+3})' = \frac{1}{2 \cdot \sqrt{e^{x+3}} \cdot \cos^2 \sqrt{e^{x+3}}} \cdot e^{x+3} \cdot (x+3)' = \\ &= \frac{e^{x+3}}{2 \cdot \sqrt{e^{x+3}} \cdot \cos^2 \sqrt{e^{x+3}}} \cdot 1 = \frac{e^{x+3}}{2 \cdot \sqrt{e^{x+3}} \cdot \cos^2 \sqrt{e^{x+3}}} \end{aligned}$$

Exponenciálne funkcie

$$(e^x)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{e^x - e^{x_0}}{x - x_0} = \lim_{x \rightarrow x_0} \frac{e^{x_0} \left(\frac{e^x}{e^{x_0}} - 1 \right)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{e^{x_0} (e^{x-x_0} - 1)}{x - x_0} = \lim_{x \rightarrow x_0} e^{x_0} = e^{x_0}$$

$$(a^x)' = (e^{x \cdot \ln a})' = e^{x \cdot \ln a} \cdot (x \cdot \ln a)' = a^x \cdot 1 \cdot \ln a = a^x \cdot \ln a$$

Logaritmické funkcie

$$(\ln x)' = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\ln x - \ln x_0}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\frac{\ln \frac{x}{x_0}}{\frac{x}{x_0}}}{\frac{x}{x_0} - 1} = \lim_{x \rightarrow x_0} \frac{1}{x_0} = \frac{1}{x_0}$$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a} \right)' = \frac{1}{\ln a} \cdot (\ln x)' = \frac{1}{x \cdot \ln a}$$

Hyperbolické funkcie

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{th} x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

$$\operatorname{cth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\operatorname{ch} x}{\operatorname{sh} x}$$

Zhrnutie

$(c)' = 0$	$(x^n)' = n \cdot x^{n-1}$
$\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}}$	$(\sqrt[n]{x})' = \frac{1}{n \cdot \sqrt[n]{x^{n-1}}}$
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$	$(\operatorname{cotg} x)' = \frac{-1}{\sin^2 x}$
$(\operatorname{arc} \sin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\operatorname{arc} \cos x)' = \frac{-1}{\sqrt{1-x^2}}$
$(\operatorname{arc} \operatorname{tg} x)' = \frac{1}{1+x^2}$	$(\operatorname{arc} \operatorname{cotg} x)' = \frac{-1}{1+x^2}$
$(e^x)' = e^x$	$(a^x)' = a^x \cdot \ln a$
$(\ln x)' = \frac{1}{x}$	$(\log_a x)' = \frac{1}{x \cdot \ln a}$
$(\operatorname{sh} x)' = \operatorname{ch} x$	$(\operatorname{ch} x)' = \operatorname{sh} x$
$(\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}$	$(\operatorname{cth} x)' = \frac{-1}{\operatorname{sh}^2 x}$
$(\operatorname{ar} \operatorname{sh} x)' = \frac{1}{\sqrt{1+x^2}}$	$(\operatorname{ar} \operatorname{ch} x)' = \frac{1}{\sqrt{x^2-1}}$
$(\operatorname{ar} \operatorname{th} x)' = \frac{1}{1-x^2}$	$(\operatorname{ar} \operatorname{cth} x)' = \frac{-1}{1-x^2}$

príklad:

$$(x \cdot \sin x)' = (x)' \cdot \sin x + x \cdot (\sin x)' = 1 \cdot \sin x + x \cdot \cos x = \sin x + x \cdot \cos x$$

$$(x^2 \cdot \operatorname{tg} x)' = (x^2)' \cdot \operatorname{tg} x + x^2 \cdot (\operatorname{tg} x)' = 2x \cdot \operatorname{tg} x + x^2 \cdot \frac{1}{\cos^2 x}$$

$$(x^3 \cdot 2^x)' = (x^3)' \cdot 2^x + x^3 \cdot (2^x)' = 3x^2 \cdot 2^x + x^3 \cdot 2^x \cdot \ln 2$$

$$\left(\frac{\ln x}{x^2}\right)' = \frac{(\ln x)' \cdot x^2 - \ln x \cdot (x^2)'}{(x^2)^2} = \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x \cdot \ln x}{x^4} = \frac{x(1 - 2 \ln x)}{x^4} = \frac{1 - 2 \ln x}{x^3}$$

$$\begin{aligned} \left(\frac{\sqrt[3]{x^4}}{e^x}\right)' &= \frac{\left(\frac{4}{x^3}\right)' \cdot e^x - \sqrt[3]{x^4} \cdot (e^x)'}{(e^x)^2} = \frac{\frac{4}{3} \cdot \frac{1}{x^3} \cdot e^x - \sqrt[3]{x^4} \cdot e^x}{(e^x)^2} = \frac{\frac{4}{3} \cdot \frac{1}{x^3} \cdot e^x - \sqrt[3]{x^4} \cdot e^x}{(e^x)^2} = \frac{\frac{4}{3} \cdot \frac{1}{x^3} \cdot e^x - \sqrt[3]{x^4} \cdot e^x}{(e^x)^2} = \frac{4 \cdot \sqrt[3]{x} \cdot e^x - 3 \cdot \sqrt[3]{x^4} \cdot e^x}{3 \cdot (e^x)^2} = \frac{e^x (4 \cdot \sqrt[3]{x} - 3 \cdot \sqrt[3]{x^4})}{3 \cdot (e^x)^2} = \\ &= \frac{4 \cdot \sqrt[3]{x} - 3 \cdot \sqrt[3]{x^4}}{3 \cdot e^x} \end{aligned}$$

$$(x^x)' = (e^{x \cdot \ln x})' = e^{x \cdot \ln x} \cdot (x \cdot \ln x)' = x^x \cdot ((x)') \cdot \ln x + x \cdot (\ln x)' = x^x \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) = x^x \cdot (\ln x + 1)$$